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Multi-Kategorien und Operaden in der Semiotik?

1. Die Definition der Multi-Kategorie in Wikipedia ist so schön, dass ich sie hier gleich reproduzieren möchte:

In mathematics (especially category theory), a **multicategory** is a generalization of the concept of category that allows morphisms of multiple arity. If morphisms in a category are viewed as analogous to functions, then morphisms in a multicategory are analogous to functions of several variables.

Definition [edit]

A multicategory consists of

- a collection (often a proper class) of objects;
- for every finite sequence $(X_1, X_2, \ldots, X_n)$ of objects (for $n := 0, 1, 2, \ldots$) and object Y, a set of *morphisms* from $X_1, X_2, \ldots$, and X_n to Y; and
- for every object X, a special identity morphism (with $n := 1$) from X to X.

Additionally, there are composition operations: Given a sequence of sequences $(X_{1,1}, X_{1,2}, \ldots, X_{1,n_1}; X_{2,1}, X_{2,2}, \ldots, X_{2,n_2}; \ldots; X_{m,1}, X_{m,2}, \ldots, X_{m,n_m})$ of objects, a sequence $(Y_1, Y_2, \ldots, Y_m)$ of objects, and an object Z: if

- f_1 is a morphism from $X_{1,1}, X_{1,2}, \ldots$, and $X_{1,n}$ to Y_1;
- f_2 is a morphism from $X_{2,1}, X_{2,2}, \ldots$, and $X_{2,n}$ to Y_2;
- …;
- f_m is a morphism from $X_{m,1}, X_{m,2}, \ldots$, and $X_{m,n}$ to Y_m; and
- g is a morphism from $Y_1, Y_2, \ldots$, and Y_m to Z:

then there is a composite morphism $g(f_1, f_2, \ldots, f_m)$ from $X_{1,1}, X_{1,2}, \ldots, X_{1,n_1}, X_{2,1}, X_{2,2}, \ldots, X_{2,n_2}, \ldots, X_{m,1}, X_{m,2}, \ldots$, and X_{m,n_m} to Z. This must satisfy certain axioms:

- If m is 1, Z is Y, and g is the identity morphism for Y, then $g(f)$ must equal f;
- if n_1 is 1, n_2 is 1, …, n_m is 1, X_1 is Y_1, X_2 is Y_2, …, X_m is Y_m, f_1 is the identity morphism for Y_1, f_2 is the identity morphism for Y_2, …, and f_m is the identity morphism for Y_m, then $g(f_1, f_2, \ldots, f_m)$ must equal g; and
- an associativity condition (involving a further level of composition) that takes a long time to write down.

Multi-Kategorien setzen offenbar die in Toth (2010) erstmals skizzierte zahlentheoretische Einführung des Zeichens voraus:

ZR+ = (X, Y, Z) = (X, σX, σσX) := {{3, ..., n}, {2, ..., m}, {1, ..., o}}.

Das kann man aber auch so darstellen:

ZR+ = {{3, ..., n}, {2, ..., m}, {1, ..., o}} = $\{\mathbb{N}\backslash\{1,2\}, \mathbb{N}\backslash\{1\}, \mathbb{N}\}$,

wobei also die Objekte $\in \mathbb{N}$ sind und die Morphismen f_n und g aus den Morphismen β° = (.)3(.) → (.)2(.) sowie α° = (.)2(.) → (.)1(.) der Peirceschen kategoriellen Semiotik redefiniert werden (vgl. z.B. Toth 1997, S. 21 ff.).

2. Nun ist der kategorielle Basisbegriff der Operaden – sozusagen Verallgemeinerungen universeller Algebren, die selbst Verallgemeinerungen „bourbakischer Strukturen" waren –, wenigstens was seine kategorietheoretische (weniger seine ursprünglich topologische) Interpretation anbetrifft, auf dem soeben eingeführten Begriff der Multi-Kategorie basiert. Auch hier zitiere ich wieder aus Wikipedia:

In category theory, an **operad without permutations** (sometimes called a **non-symmetric**, **non-Σ** or **plain** operad) is a multicategory with one object. More explicitly, such an operad consists of the following:

- a sequence $(P(n))_{n\in\mathbb{N}}$ of sets, whose elements are called *n-ary operations*,
- for each integers n, k_1, ..., k_n a function

$$\begin{array}{ccc} P(n)\times P(k_1)\times\cdots\times P(k_n) & \to & P(k_1+\cdots+k_n) \\ (\theta,\theta_1,\ldots,\theta_n) & \mapsto & \theta\circ(\theta_1,\ldots,\theta_n) \end{array}$$

called *composition*,

- an element 1 in $P(1)$ called the *identity*,

satisfying the following coherence properties:

- *associativity*:

$$\theta\circ(\theta_1\circ(\theta_{1,1},\ldots,\theta_{1,k_1}),\ldots,\theta_n\circ(\theta_{n,1},\ldots,\theta_{n,k_n}))=(\theta\circ(\theta_1,\ldots,\theta_n))\circ(\theta_{1,1},\ldots,\theta_{1,k_1},\ldots,\theta_{n,1},\ldots,\theta_{n,k_n})$$

- *identity*:

$$\theta\circ(1,\ldots,1)=\theta=1\circ\theta$$

(where the number of arguments correspond to the arities of the operations).

Die Familie der $P(n)_{n\in\mathbb{N}}$ sind in der Semiotik einfach wieder die drei Mengen $\{\mathbb{N}\backslash\{1,2\}, \mathbb{N}\backslash\{1\}, \mathbb{N}\}$. Die allgemein wie folgt definierten Morphismen:

A **morphism of operads** $f : P \to Q$ consists of a sequence

$$(f_n : P(n) \to Q(n))_{n\in\mathbb{N}}$$

which:

- preserves composition: for every *n*-ary operation θ and operations θ_1, ..., θ_n,

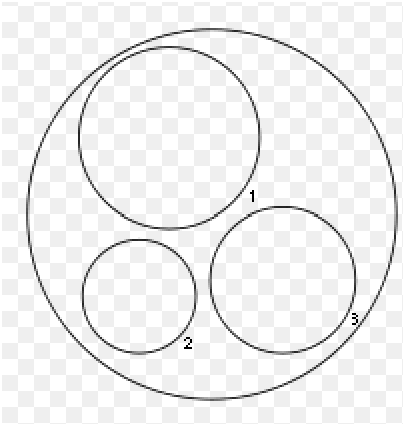
$$f(\theta\circ(\theta_1,\ldots,\theta_n))=f(\theta)\circ(f(\theta_1),\ldots,f(\theta_n))$$

- preserves identity:

$f(1) = 1$.

Operads were originally defined topologically, by May, but his full definition requires symmetric group actions on the $P(n)$ that are suitably related to the maps θ_n. The permutation actions are additional structure that is vital to the original and most later applications.

sind genau gleich zu behandeln wie oben für allgemeine Multi-Kategorien bestimmt. Ein Problem sehe ich jedoch bei einer topologisch-semiotischen Einführung von Operaden, insofern in dem folgenden Scheiben-Modell von Markl, Shnider and Stasheff (2002) die Bedingung der „disjointness" der „Little-Something"-Operaden bestehen muss:



(aus: Markl, Shnider, Stasheff 2002),

da ja auch im zahlentheoretischen Zeichenmodell die Verschachtelung der monadischen, dyadischen und triadischen Relationen natürlich beibehalten ist.

Bibliographie

Markl, Martin/Snider, Steve/Stasheff, Jim, Operads in Algebra, Topology, and Physics. Univ. of North Carolina Press 2002

Toth, Alfred, Semiotisch-Relationale Grammatik. Tübingen 1997

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